Math 3280 Tutorial 9

Recall :

① Conditional probability

1. X, Y are discrete r.v.s.

$$P_{X|Y}(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}, \quad P_Y(y) > 0$$

2. X, Y are jointly cts. the conditional density, given $Y=y$, is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad f_Y(y) > 0.$$

Example 1:

Suppose X, Y are integer-valued r.v.s. Let

$$P(i|j) = P(X=i \mid Y=j)$$
$$q(j|i) = P(Y=j \mid X=i)$$

Show that

$$P(X=i, Y=j) = \frac{P(i|j)}{\sum_i \frac{P(i|j)}{q(j|i)}}.$$

Solution:

$$P(i|j) = P(X=i \mid Y=j) = \frac{P(X=i, Y=j)}{P(Y=j)}$$

$$\Rightarrow P(X=i, Y=j) = P(Y=j) \cdot P(i|j)$$

$$= P(i|j) \Big/ \left(\frac{1}{P(Y=j)}\right)$$

$$\left(\sum_i P(X=i) = 1\right)$$

$$= P(i|j) \Big/ \left(\frac{\sum_i P(Y=i)}{P(Y=j)}\right)$$

$$\left(P(\cdot|Y=j)\right)$$

$$P(X=i,Y=j)/P(Y=j) = P(i|j)$$

$$P(X=i,Y=j)/P(X=j) = q(j|i)$$

$$= \frac{P(i|j)}{\left(\sum_i \frac{P(X=i,Y=j)/P(Y=j)}{P(X=i,Y=j)/P(X=j)}\right)} = \frac{P(i|j)}{\sum_i \frac{P(i|j)}{q(j|i)}}$$

Recal: ② X, Y are joint cts r.v.s. with density $f_{X,Y}(x,y)$.

$Y_1 = g_1(X,Y)$. $Y_2 = g_2(X,Y)$

If X, Y can be solved in terms of Y_1, Y_2 and $g_1 g_2$ have cts partial derivative.

$$J = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \frac{\partial g_1}{\partial x} \cdot \frac{\partial g_2}{\partial y} - \frac{\partial g_1}{\partial y} \cdot \frac{\partial g_2}{\partial x} \neq 0.$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2) \cdot |J|^{-1}.$$

Example 2: Let X, Y, Z be independent standard normal r.v.s.

$U = X+Y+Z$, $V = X-Y$, $W = X-Z$.

Find the joint distribution of U, V, W.

Solution:

$$J = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix}$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= 3$$

$$\begin{cases} U = X+Y+Z \\ V = X-Y \\ W = X-Z \end{cases} \Rightarrow \begin{cases} X = \frac{u+v+w}{3} \\ Y = \frac{u-2v+w}{3} \\ Z = \frac{u+v-2w}{3} \end{cases}$$

$$f_{X,Y,Z}(x,y,z) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right) \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right)$$

Then the joint density function of U, V, W is

$$f_{U,V,W}(u,v,w) = |J|^{-1} \cdot f_{X,Y,Z}(x,y,z)$$

$$= \frac{1}{3} \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^3 \cdot e^{-\frac{x^2+y^2+z^2}{2}}$$

$$= \frac{1}{3} \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^3 \cdot e^{-\frac{\left(\frac{u+v+w}{3}\right)^2 + \left(\frac{u-2v+w}{3}\right)^2 + \left(\frac{u+v-2w}{3}\right)^2}{2}}$$

$$= \frac{1}{3} \frac{1}{(\sqrt{2\pi})^3} e^{-\left(\frac{u^2}{6} + \frac{v^2}{3} + \frac{w^2}{3} - \frac{vw}{3}\right)}$$

Example 3. X_1, X_2 be two independent exponential r.v.s, each having parameter λ, find the joint density of

$$Y_1 = X_1 + X_2,$$
$$Y_2 = e^{X_1}$$

Solution: The Jacobian of $Y_1 = X_1 + X_2$, $Y_2 = e^{X_1}$ is

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ e^{x_1} & 0 \end{vmatrix} = \underline{-e^{x_1}}.$$

$$\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = e^{X_1} \end{cases} \Rightarrow \begin{cases} X_1 = \ln(Y_2) \\ X_2 = Y_1 - \ln Y_2 \end{cases},$$

Since X_1, X_2 are exponential r.v.s,
we have $X_1 \geq 0$, $X_2 \geq 0$.

$$\Rightarrow \begin{matrix} \ln(Y_2) \geq 0 \\ Y_1 - \ln Y_2 \geq 0 \end{matrix} \Rightarrow \begin{matrix} Y_2 \geq 1. \\ e^{y_1} \geq y_2 \end{matrix} \Rightarrow e^{y_1} \geq y_2 \geq 1$$

$$f_{Y_1, Y_2}(y_1, y_2) = |J|^{-1} \cdot f_{X_1, X_2}(x_1, x_2) \quad (X_1, X_2 \sim \exp(\lambda))$$

independent

$$= e^{-x_1} \cdot (\lambda e^{-\lambda x_1}) \cdot (\lambda) e^{-\lambda x_2})$$
$$= e^{-\ln y_2} \cdot \lambda^2 \cdot e^{-\lambda \ln y_2} \cdot e^{-\lambda(y_1 - \ln y_2)}$$
$$= \frac{\lambda^2}{y_2} \cdot e^{-\lambda y_1} \quad (1 \leq y_2 \leq e^{y_1}).$$

Example 4. Let X, Y, Z be independent r.v.s with identical density function

$$f(x)=\begin{cases} e^{-x}, & x>0 \\ 0, & o.w \end{cases}$$

Find the joint density function of

$U = X+Y,$

$V = X+Z,$

$W = Y+Z.$

Solution: The Jacobian of U, V, W is given by

$$|J| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2.$$

$$\begin{cases} U = X+Y \\ V = X+Z \\ W = Y+Z \end{cases} \Rightarrow \begin{cases} X = \frac{U+V-W}{2} > 0, \\ Y = \frac{U+W-V}{2} > 0 \\ Z = \frac{V+W-U}{2} > 0 \end{cases}$$

$U+V>W,$
$U+W>V$
$V+W>U.$

The density function of X, Y, Z

$$f_{X,Y,Z}(x,y,z) = e^{-(x+y+z)}, \quad x>0, y>0, z>0,$$

Then joint density function of U, V, W is

$$f_{U,V,W}(u,v,w) = |J|^{-1} \cdot f_{X,Y,Z}(x,y,z)$$

$$= \frac{1}{2} \cdot e^{-(x+y+z)}$$

$$= \frac{1}{2} \cdot e^{-\left(\frac{u+v-w}{2} + \frac{u+w-v}{2} + \frac{v+w-u}{2}\right)}$$

$$= \frac{1}{2} \cdot e^{-\frac{u+v+w}{2}}$$

$u+v>w$

$u+w>v$

$v+w>u$